E content for students of patliputra university

B. Sc. (Honrs) Part 1Paper 2

Subject Mathematics

Title/Heading of topic : Reduction formula

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3.1 INTRODUCTION

In the first two units of this block we have introduced the concept of a definite integral and have obtained the values of integrals of some standard forms. We have also studied two important methods of evaluating integrals, namely, the method of substitution and the method of integration by parts. In the solution of many physical or engineering problems, we have to integrate some integrands involving powers or products of trigonometric functions. In this unit we shall devise a quicker method for evaluating these integrals. We shall consider some standard forms of integrands one by one, and derive formulas to integrate them.

The integrands which we will discuss here have one thing in common. They depend upon an integer parameter. By using the method of integration by parts we shall try to express such an integral in terms of another similar integral with a lower value of the parameter. You will see that by the repeated use of this technique, we shall be able to evaluate the given integral.

Objectives

After reading this unit you should be able to derive and apply the reduction formulas for

- $\int x^n e^x dx$
- $\int \sin^n x dx, \int \cos^n x\, dx\ \int \tan^n x dx$, etc.
- $\int \sin^m x \cos^n x dx$
- $\int e^{ax} \sin^n x\, dx$
- $\int \sinh^n x dx, \int \cosh^n x dx$

3.2 REDUCTION FORMULA

Sometimes the integrand is not only a function of the independent variable, but also depends upon a number n (usually an integer). For example, in $\int \sin^n x dx$, the integrand $\sin^n x$ depends on x and n. Similarly, in $\int e^x \cos mx\, dx$, the integrand $e^x \cos mx$ depends on x and m. The numbers n and m in these two examples are called parameters. We shall discuss only integer parameter here.

we shall study in this section.

3.4.1 Integrand of the Type $\sin^m x \cos^n x$

The function $\sin^m x\cos^n x$ depends on two parameters m and n. To find a reduction formula for

$\int \sin^m x \cos^n dx$, let us first write

$$I_{m,n} = \int \sin^m x \cos^n x dx$$

Since we have two parameters here, we shall take a reduction formula to mean a formula connecting $I_{m,n}$ and $I_{p,q}$, where either $p < m$, or $q < n$, or both $p < m$, $q < n$ hold. In other words, the value of at least one parameter should be reduced.

$$\text{If } n = 1 \; I_{m,1} = \int \sin^m x \cos x \, dx \; \begin{cases} \dfrac{\sin^{m+1} x}{m+1} + c, \text{ when } m \neq -1 \\ \ln|\sin x| + c, \text{ when } m = -1 \end{cases}$$

Hence we assume that $n > 1$. Now,

$$I_{m,n} = \int \sin^m x \cos^n x \, dx = \int \cos^{n-1} x \, (\sin^m x \cos x) \, dx$$

Integrating by parts we get

$$I_{m,n} = \frac{\cos^{n-1} x.\sin^{m+1} x}{m+1} - \int (n-1)\cos^{n-2} x \, (-\sin x) \frac{\sin^{m+1} x}{m+1} dx, \text{ if } m \neq -1$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x \, (1 - \cos^2 x) \, dx$$

$$= \frac{\cos^{n-1} x.\sin^{m+1} x}{m+1} + \frac{n-1}{m+1} [I_{m,n-2} - I_{m,n}]$$

Therefore,

$$I_{m,n} + \frac{n-1}{m+1} I_{m,n} = \frac{m+n}{m+1}, I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

This gives us,

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1n} + \frac{n-1}{m+n} I_{m,n-2} \quad \text{.............. (3)}$$

Remember we have taken $n > 1$

But, surely this formula will not work if $m + n = 0$. So, what do we do if $m + n = 0$? Actually we have a simple way out. If $m + n = 0$, then since, n is positive, we write $m = -n$.

Hence $I_{-n,n} = \int \sin^{-n} x \cos^n x dx = \int \cot^n x dx$, which is easy to evaluate using the reduction formula derived in Sec. 3 (See E2)).

To obtain formula (3) we had started with the assumption that $n > 1$.
Instead of this, if we assume that $m > 1$, we can write

$I_{m,n} = \int \sin^m x \cos^n x dx = \int \sin^{m-1} x \, (\cos^n x \sin x) \, dx$. Integrating this by parts we get

$$I_{m,n} = \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} - (m-1) \int \sin^{m-2} x \cos x \frac{(-\cos^{n+1} x)}{n+1} dx \text{ for } n \neq -1.$$

3.3.1 Reduction Formulas for $\int \sin^n x\, dx$ and $\int \cos^n x\, dx$

In this sub-section we will consider integrands which are powers of either sinx or cosx. Let's take a power of sinx first. For evaluating $\int \sin^n x dx$, we write

$$I^n = \int \sin^n x dx = \int \sin^{n-1} x \sin x dx, \text{ if } n > 1.$$

Taking $\sin^{n-1}x$ as the first function and sin x as the second and integrating by parts, we get

$$I_n = -\sin^{n-1} x \cos x - (n-1) \int \sin^{n-2} x \cos x\,(-\cos x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) [\int \sin^{n-2} x(1-\sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) [\int \sin^{n-2} x\, dx - \int \sin^n x)\, dx]$$

$$= -\sin^{n-1} x \cos x + (n-1) [I_{n-2} - I_n]$$

Hence,

$$I_n + (n-1) I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

That is, $nI_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$ Or

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

This is the reduction formula for $\int \sin^n x dx$ (valid for $n \geq 2$).

for $n = 1$

$\int \sin^n x dx = \int \sin x\, dx$

$-\cos x + c$

Example 2 We will now use the reduction formula for $\int \sin^n x dx$ to evaluate the definite integral, $\int_0^{\pi/2} \sin^5 x dx$. We first observe that

$$\int_0^{\pi/2} \sin^n x dx = \left.\frac{-\sin^{n-1} x \cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx$$

$$= \frac{n-1}{n} \int^{\pi/2} \sin^{n-2} x dx,\ n \geq 2.$$

Thus, $\int_0^{\pi/2} \sin^5 x dx = \frac{4}{5} \int_0^{\pi/2} \sin^3 x dx$

$$= \frac{4}{5} \cdot \frac{2}{3} \int_0^{\pi/2} \sin x dx$$

$$= \frac{8}{15} \left.(-\cos x)\right]_0^{\pi/2}$$

$$= \frac{8}{15}$$

Let us now derive the reduction formula for $\int \cos^n x dx$. Again let us write

$$I_n = \int \cos^n x dx = \int \cos^{n-1} x \cos x dx,\ n > 1.$$

Integrating this integral by parts we get

$$I_n = \int \cos^{n-1} x \sin x - \int (n-1) \cos^{n-2} x\,(-\sin x).\ \sin x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x(1-\cos^2 x) dx$$

$$= \cos^{n-1} x \sin x + (n-1) (I_{n-2} - I_n)$$

By rearranging the terms we get

$$I_n = \int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$$

This formula is valid for $n \geq 2$. What happens when $n = 0$ or 1? You will agree that the integral in each case is easy to evaluate.
As we have observed in Example 2,

$$\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx, \; n \geq 2.$$

Using this formula repreatedly we get

$$\int_0^{\pi/2} \sin^n x dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{4}{5} \cdot \frac{2}{3} \int_0^{\pi/2} \sin x \, dx, \text{ if } n \text{ is an odd number, } n \geq 3. \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{3}{4} \cdot \frac{1}{2} \int_0^{\pi/2} dx, \text{ if } n \text{ is an even number, } n \geq 2. \end{cases}$$

This means

$$\int_0^{\pi/2} \sin^n x dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots\cdots \frac{4}{5} \cdot \frac{2}{3} \text{ if } n \text{ is odd, and } n \geq 3 \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots\cdots \frac{3}{4} \cdot \frac{1}{2} \frac{\pi}{2}, \text{ if } n \text{ is even } n \geq 2 \end{cases}$$

We can reverse the order of the factors, and write this as

$$\int_0^{\pi/2} \sin^n x dx = \begin{cases} \frac{2}{3} \frac{4}{5} \cdots\cdots \frac{n-3}{n-2} \cdot \frac{n-1}{n}. \text{ if } n \text{ is odd, } n \geq 3 \\ \frac{1}{2} \frac{3}{4} \cdots\cdots \frac{n-1}{n} \cdot \frac{\pi}{2}. \text{ if } n \text{ is even, } n \geq 2 \end{cases}$$

Arguing similarly for $\int_0^{\pi/2} \cos^n x dx$ we get

$$\int_0^{\pi/2} \cos^n x dx = \int_0^{\pi/2} \sin^n x \, dx = \begin{cases} \frac{2}{3} \cdot \frac{4}{5} \cdots\cdots \frac{n-1}{n}. \text{if } n \text{ is odd, and } n \geq 3 \\ \frac{1}{2} \frac{3}{4} \cdots\cdots \frac{n-1}{n} \cdot \frac{\pi}{2} \text{ if } n \text{ is even, } n \geq 2 \end{cases}$$

We are leaving the proof of this formula to you as an exercise See E1)

E E1) Prove that

$$\int_0^{\pi/2} \cos^n x dx = \begin{cases} \frac{2}{3} \cdot \frac{4}{5} \cdots\cdots \frac{n-1}{n}, \text{ if } n \text{ is odd, } n \geq 3 \\ \frac{1}{2} \frac{3}{4} \cdots\cdots \frac{n-1}{n} \cdot \cdot \frac{\pi}{2}, \text{ if } n \text{ is even, } n \geq 2 \end{cases}$$

3.3.2 Reduction Formulas for $\int \tan^n x dx$ and $\int \sec^n x dx$

In this sub-section we will take up two other trigonometric functions : tanx and secx. This is, we will derive the reduction formulas for $\int \tan^n x dx$ and $\int \sec^n x dx$. To derive a reduction formula for $\int \tan^n x dx$, $n > 2$. we start in a slightly different manner. Instead of writing $\tan^n x = \tan x \tan^{n-1} x$, as we did in the case of $\sin^n x$ and $\cos^n x$,. we shall write $\tan^n x = \tan^{n-2} x \tan^2 x$. You will shortly see the reason behind this. So we write

$$I_n = \int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx.$$

$$= \int \tan^{n-2} x (\sec^2 x - 1) dx$$

$$= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx \quad \text{---(2)}$$

You must have observed that the second integral on the right hand side is I_{n-2}. Now in the first integral on the right hand side, the integrand is of the form $[f(x)]^m . f'(x)$

As we have seen in Unit 11,

$$\int [f(x)]^m f'(x) dx = \frac{[f(x)]^{m+1}}{m+1} + c$$

Thus, $\int \tan^{n-2} x \sec^2 x dx = \frac{\tan^{n-1} x}{n-1} + c$

Therefore, (2) give $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$.

Thus the reduction formula for $\int \tan^n x \, dx$ is

$$\int \tan^n x dx = I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}.$$

You must have now realised why we wrote $\tan^n x = \tan^2 x \tan^{n-2} x$

To derive the reduction formula for $\int \sec^n x dx$ $(n > 2)$, we first write $\sec^n x = \sec^{n-2} x \sec^2 x$,

and then integrate by parts. Thus

$$I_n = \int \sec^n x dx = \int \sec^{n-2} x \sec^2 x dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-3} x, \sec x \tan^2 x dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) dx$$

$$= \sec^{n-2} x \tan x - (n-2)(I_n - I_{n-2})$$

After rearranging the terms we get

$$\int \sec^n x dx = I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

These formulas for $\int \tan^n x dx$ and $\int \sec^n x dx$ are valid for $n > 2$. For $n = 0$, 1 and 2, the integrals $\int \tan^n x dx$ and $\int \sec^n x dx$ can be easily evaluated. You have come across them in Units 1 and 2.

Example 3 Let's calculate i) $\int_0^{\pi/4} \tan^5 x dx$ and ii) $\int_0^{\pi/4} \sec^6 x \, dx$

i) $$\int_0^{\pi/4} \tan^5 x dx = \frac{\tan^4 x}{4}\Big]_0^{\pi/4} - \int_0^{\pi/4} \tan^3 x dx$$

$$= \frac{1}{4} - \frac{\tan^2 x}{x}\Big]_0^{\pi/4} + \int_0^{\pi/4} \tan x dx$$

$$= \frac{1}{4} - \frac{1}{2} + \int_0^{\pi/4} \frac{\sin x}{\cos x} dx$$

$$= -\frac{1}{4} - \ln(\cos x)\Big]_0^{\pi/4}$$

$$= -\frac{1}{4} - \ln\frac{1}{\sqrt{2}} + \ln 1$$

$$= -\frac{1}{4} \ln \sqrt{2}$$

ii) $$\int_0^{\pi/4} \sec^6 x dx = \frac{\sec^4 x \tan x}{5}\Big]_0^{\pi/4} + \frac{4}{5} \int_0^{\pi/4} \sec^4 x dx$$

$$= \frac{4}{5} + \frac{4}{5}\left\{\frac{\sec^2 x \tan x}{3}\Big]_0^{\pi/4} + \frac{2}{3}\int_0^{\pi/4} \sec^2 x dx\right\}$$

$$= \frac{4}{5} + \frac{8}{15} + \frac{8}{15} \int_0^{\pi/4} \sec^2 x dx$$

$$= \frac{4}{3} + \frac{8}{15} \tan x \Big]_0^{\pi/4} = \frac{28}{15}$$

On the basis of our discussion in this section you will be able to solve these exercises.

3.4 INTEGRALS INVOLVING PRODUCTS OF TRIGONOMETRIC FUNCITONS

In the last section we have seen the reduction formulas for the case where integrands were powers of a single trigonometric function. Here we shall consider some integrands involving products of powers of trigonometric functions. The technique of finding a reduction formula basically involves integration by parts. Since there can be more than one way of writing the integrand as a product of two functions, you will see that we can have many reduction formulas for the same integral. We start with the first one of the two types of integrands which we shall study in this section.

3.4.1 Integrand of the Type $\sin^m x \cos^n x$

The function $\sin^m x\cos^n x$ depends on two parameters m and n. To find a reduction formula for

$\int \sin^m x \cos^n dx$, let us first write

$$I_{m,n} = \int \sin^m x \cos^n x dx$$

Since we have two parameters here, we shall take a reduction formula to mean a formula connecting $I_{m,n}$ and $I_{p,q}$, where either $p < m$, or $q < n$, or both $p < m$, $q < n$ hold. In other words, the value of at least one parameter should be reduced.

$$\text{If } n=1 \; I_{m,1} = \int \sin^m x \cos x \, dx \begin{cases} \dfrac{\sin^{m+1} x}{m+1} + c, \text{ when } m \neq -1 \\ \ln|\sin x| + c, \text{ when } m = -1 \end{cases}$$

Hence we assume that $n > 1$. Now,

$$I_{m,n} = \int \sin^m x \cos^n x \, dx = \int \cos^{n-1} x \, (\sin^m x \cos x) \, dx$$

Integrating by parts we get

$$I_{m,n} = \frac{\cos^{n-1} x, . \sin^{m+1} x}{m+1} - \int (n-1)\cos^{n-2} x \, (-\sin x) \frac{\sin^{m+1} x}{m+1} dx, \text{ if } m \neq -1$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x \, (1-\cos^2 x) \, dx$$

$$= \frac{\cos^{n-1} x . \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} [I_{m,n-2} - I_{m,n}]$$

Therefore,

$$I_{m,n} + \frac{n-1}{m+1} I_{m,n} = \frac{m+n}{m+1}, I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

This gives us,

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1n} + \frac{n-1}{m+n} I_{m,n-2} \quad \text{............(3)}$$

Remember we have taken $n > 1$

But, surely this formula will not work if $m+n = 0$. So, what do we do if $m+n=0$? Actually we have a simple way out. If $m+n=0$, then since, n is positive, we write $m=-n$.

Hence $I_{-n,n} = \int \sin^{-n} x \cos^n x dx = \int \cot^n x dx$, which is easy to evaluate using the reduction formula derived in Sec. 3 (See E2)).

To obtain formula (3) we had started with the assumption that $n > 1$.
Instead of this, if we assume that $m > 1$, we can write

$I_{m,n} = \int \sin^m x \cos^n x dx = \int \sin^{m-1} x \, (\cos^n x \sin x) \, dx$. Integrating this by parts we get

$$I_{m,n} = \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} - (m-1) \int \sin^{m-2} x \cos x \frac{(-\cos^{n+1} x)}{n+1} dx \text{ for } n \neq -1.$$

$$= \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^n x\,(1-\sin^2 x)\,dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}(I_{m-2,n} - I_{m,n})$$

From this we obtain

$$I_{m,n} = \frac{-\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}(I_{m-2,n}) \text{ ....(4)}$$

If m or n is a positive odd integer, we can proceed as follows :

Suppose $n = 2p + 1$ $P > 0$, then

$$I_{m,n} = \int \sin^m x \cos^{2p+1} x dx = \int \sin^m x\,(1-\sin^2 x)^p \cos x\,dx$$

$$= \int t^m (1-t^2)^p dt \text{ we put } t = \sin x.$$

Expanding $(1-t^2)^p$ by binomial theorem and integrating term by term, we get

$$I_{m,n} = \frac{t^{m+1}}{m+1} - c(p,1)\frac{t^{m+3}}{m+3} - C(p,2)\frac{t^{m+5}}{m+5} \ldots\ldots + \frac{(-1)^p t^{m+2p+1}}{m+2p+1} + c$$

$$= \frac{\sin^{m+1} x}{m+1} - C(p,1)\frac{\sin^{m+3} x}{m+3} + C(p,2)\frac{\sin^{m+5} x}{m+5} - \ldots\ldots\ldots + \frac{(-1)^n \sin^{m+2p+1} x}{m+2p+1} + c$$

If m and n are positive integers, by repeated applications of formula (3) or formula (4), we keep reducing n or m by 2 at each step. Thus, eventually, we come to an integral of the form $I_{m,0}$ or $I_{m,1}$ or $I_{1,n}$ or $I_{0,n}$. In the previous section we have seen how these can be evaluated. This means we should be able to evaluate $I_{m,n}$ in a finite number of steps. We shall now look at an example to see how these formulas are used.

Example 4 Let us evaluate $\int_0^{\pi/2} \sin^4 x \cos^6 x\,dx$. Here $m = 4$ and $n = 6$. Since m is the smaller of the two, we shall employ formula (4) which reduces m at each step.

$$\int_0^{\pi/2} \sin^4 x \cos^6 x dx = \left.\frac{-\sin^3 x \cos^7 x}{10}\right]_0^{\pi/2} + \frac{3}{10}\int_0^{\pi/2} \sin^2 x \cos^6 x\,dx$$

$$= \frac{3}{10}\int_0^{\pi/2} \sin^2 x \cos^6 x dx$$

$$= \frac{3}{10}\left\{\left.\frac{-\sin x \cos^7 x}{8}\right]_0^{\pi/2} + \frac{1}{8}\int_0^{\pi/2} \cos^6 x dx\right\},$$

using formula (4) again.

$$= \frac{3}{80}\int_0^{\pi/2} \cos^6 x dx$$

$$= \frac{3}{80} \times \frac{15\pi}{96} \text{ (from E 2)b))} = \frac{3\pi}{512}$$

Are you ready to solve some exercises now?

E E5) In deriving formula (4) we had assumed that $m > 1$. How would you evaluate, $I_{m,n}$ if $m = 1$?

3.4.2 Integrand of the Type $e^{ax}\sin^n x$

In this sub-section we will consider the evaluation of those integrals, where the integrand is a product of a power of a trigonometric function and an exponential function. That is, we will consider integrands of the type $e^{ax}\sin^n x$. Let us denote $\int e^{ax}\sin^n x\,dx$ by L_n, and integrate it by parts, taking $\sin^n x$ as the first function and e^{ax} as the second function. This gives us

$$L_n = \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\int e^{ax}\sin^{n-1}x\cos x\,dx.$$

We shall now evaluate the integral on the right hand side, again by parts, with $\sin^{n-1}x\cos x$ as the first function and e^{ax} as the second one. Thus,

$$L_n = \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\left[\frac{e^{ax}\sin^{n-1}x\cos x}{a} - \frac{1}{a}\int e^{ax}\{(n-1)\sin^{n-2}x\cos^2 x - \sin^n x\}\,dx\right]$$

$$= \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\left[\frac{e^{ax}\sin^{n-1}x\cos x}{a} - \frac{1}{a}\int e^{ax}\{(n-1)\sin^{n-2}x - n\sin^n x\}dx\right]$$

$$(n-1)\sin^{n-2}x\cos^2 x$$
$$= (n-1)\sin^{n-2}x\,(1-\sin^2 x)$$
$$= (n-1)\sin^{n-2}x - (n-1)\sin^n x$$

This means

$$L_n = \frac{e^{ax}\sin^n x}{a} - \frac{ne^{ax}\sin^{n-1}x\cos x}{a^2} + \frac{n(n-1)}{a^2}L_{n-2} - \frac{n^2}{a^2}L_n$$

Rearranging the terms we get

$$L_n = \frac{ae^{ax}\sin^n x}{n^2+a^2} - \frac{ne^{ax}\sin^{n-1} x\cos x}{n^2+a^2} + \frac{n(n-1)}{n^2+a^2}L_{n-2}.$$

Given any L_n, we use this reduction formula repeatedly, till we get L_1 or L_0 (depending on whether n is odd or even). Since L_1 and L_0 are easy to evaluate, we are sure you can evaluate them yourself. (See E8)). This means that L_n can be evaluated for any positive integer n.

Remark 1 If we put a = 0 in L_n, it reduces to the integral $\int \sin^n x dx$. This suggests that the reduction formula for $\int \sin^n x dx$ which we have derived in Sec. 3 is a special case of the reduction formula for L_n.

If you have followed the arguments in this sub-section closely, you should be able to do the exercises below.

E E8) Prove that

a) $L_0 = \frac{e^{ax}}{a} + c$ b) $L_1 = \int e^{ax}\sin x dx = \frac{e^{ax}}{1+a^2}(a\sin x - \cos x) + c.$

3.5 INTEGRALS INVOLVING HYPERBOLIC FUNCTIONS

In this section we shall discuss the evaluation of integrals of the type

$\int \sinh^n x dx$, $\int \cosh^n x dx$, etc.

Actually, you will find that the evaluation of these integrals does not involve any new techniques. In fact, the procedure we follow here is very similar to the one we followed for integrating $\sin^n x$, $\cos^n x$ etc. Let us find the reduction formula for, say $\int \tanh^n x dx$. We are sure you will be able to follow this easily and derive the reduction formulas for the other hyperbolic functions (see E11)).

If $I_n = \int \tanh^n x dx$, we can write

$$I_n = \int \tanh^{n-2} x \tanh^2 x \, dx$$

$$= \int \tanh^{n-2} x \, (1 - \operatorname{sech}^2 x) \, dx$$

$$= \int \tanh^{n-2} x dx - \int \tanh^{n-2} x \operatorname{sech}^2 x dx$$

$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$

$$= I_{n-2} - \frac{\tanh^{n-1} x}{n-1}$$

Don't you agree that the above method is similar to the one adopted for $\int \tan^n xdx$? The following exercises can be easily done now.

E E 11) Prove the following reduction formula :

$$\int \sinh^n xdx = \frac{\sinh^{n-1} x \cosh x}{n} - \frac{n-1}{n} \int \sinh^{n-2} xdx$$

E E12) Derive a reduction formula for $\int \cosh^n xdx$

That brings us to the end of this unit. We shall now summarise what we have covered in it.

3.6 SUMMARY

A reduction formula is one which links an integral dependent on a parameter with a similar integral with a lower value of the parameter.
In this unit we have derived a number of reduction formulas.

1. $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x \, dx$

2. $\int \sin^n x \, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} xdx, \; n \geq 2$

3. $\int \cos^n x \, dx = \frac{+\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx, \; n \geq 2$

4. $\int \tan^n x \, dx = \frac{-\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx, \; n > 2$

5. $\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx, \ n > 2$

6. $\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \begin{cases} \frac{2}{3} \frac{4}{5} \ldots\ldots \frac{n-1}{n}, \text{ if n is odd, } n \geq 3. \\ \frac{1}{2} \frac{3}{4} \ldots\ldots \frac{n-1}{n} \frac{\pi}{2}, \text{ if n is even, } n \geq 2. \end{cases}$

7. $\int \sin^m x \cos^n x dx = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x dx, \ n > 1$

$= \frac{-\sin^{m-1} x \cos^{n-1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x dx, \ m > 1$

8. $\int e^{ax} \sin^n x dx = \frac{ae^{ax} \sin^n x}{n^2 + a^2} - \frac{ne^{ax} \sin^{n-1} x \cos x}{n^2 + a^2} + \frac{n(n-1)}{n^2 + a^2} \int e^{ax} \sin^{n-2} dx$

9. $\int \tanh^n x dx = \frac{-\tanh^{n-1} x}{n-1} + \int \tanh^{n-2} x dx$

We have noted that the primary technique of deriving reduction formulas involves integration by parts. We have also observed that many more reduction formulas involving other trigonometric and hyperbolic functions can be derived using the same technique.

3.7 SOLUTIONS AND ANSWERS

E1) We have $\int_0^{\pi/2} \cos^n x dx = \left. \frac{\cos^{n-1} x \sin x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x dx, \ n \geq 2$

$= \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x dx$

$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \int_0^{\pi/2} \cos^{n-4} x dx$

$= \begin{cases} \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \ldots\ldots \frac{3}{4} \cdot \frac{1}{2} \int_0^{\pi/2} \cos^0 x dx, \text{ if n is even} \\ \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \ldots\ldots \frac{4}{5} \cdot \frac{2}{3} \int_0^{\pi/2} \cos x dx, \text{ if n is odd} \end{cases}$

$= \begin{cases} \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \ldots\ldots \frac{3}{4} \cdot \frac{1}{2} \frac{\pi}{2} \text{ if n is even} \\ \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \ldots\ldots \frac{4}{5} \cdot \frac{2}{3} \text{ if n is odd} \end{cases}$

E2) a) $\int_0^{\pi/2} \cos^5 x dx = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$

b) $\int_0^{\pi/2} \cos^6 x dx = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5\pi}{32}$

E3) $I_n = \int \cot^n x dx = \int \cot^{n-2} x (\operatorname{cosec}^2 x - 1) dx \ n > 2$

$= \int \cot^{n-2} \operatorname{cosec}^2 x dx - I_{n-2}$

$$\text{Therefore, } I_n = \frac{-\cot^{n-1} x}{n-1} - I_{n-2}$$

b) $$I_n = \int \operatorname{cosec}^n x\,dx = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x\,dx \quad n > 2$$

$$= -\operatorname{cosec}^{n-2} x \cot x - \int (n-2) \operatorname{cosec}^{n-2} x \cot^2 x\,dx$$

$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2) \int \operatorname{cosec}^{n-2} x (\operatorname{cosec}^2 x - 1)\,dx$$

$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2) I_n + (n-2) I_{n-2}$$

$$I_n = \frac{-\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

E4) $$\int_{\pi/4}^{\pi/2} \operatorname{cosec}^3 x\,dx = \left.\frac{-\operatorname{cosec} x \cot x}{2}\right]_{\pi/4}^{\pi/2} + \frac{1}{2}\int_{\pi/4}^{\pi/2} \operatorname{cosec} x\,dx$$

$$= -\frac{1}{\sqrt{2}} + \frac{1}{2} \ln \tan \frac{x}{2} \Big]_{\pi/4}^{\pi/2}$$

$$= -\frac{1}{\sqrt{2}} + \frac{1}{2}\left(\ln 1 - \ln \tan \frac{\pi}{8}\right)$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{2} \ln \tan \frac{\pi}{8}$$

b) $$\int_0^{\pi/2} \sin^8 x\,dx = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{35\pi}{256}$$

c) $$\int \sec^3 \theta\,d\theta = \frac{\sec\theta \tan\theta}{2} + \frac{1}{2}\int \sec\theta\,d\theta$$

$$= \frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\ln(\sec x + \tan x) + c$$

E5) If $m = 1$, $I_{m,n} = I_{1,n} = \int \sin x \cos^n x\,dx$

$$= \begin{cases} \dfrac{\cos^{n+1} x}{n+1} + c \text{ if } n \neq -1 \\ -\ln|\cos x| + c \text{ if } n = -1 \end{cases}$$

E6) $m + n = 0 \Rightarrow n = -m \Rightarrow m$ is a positive integer.

$$I_{m,n} = \int \sin^m x \cos^{-m} x\,dx = \int \frac{\sin^m x}{\cos^m x}\,dx = \int \tan^m x\,dx$$

Now use the formula for $\int \tan^m x\,dx$

E7) a) $$\int_0^{\pi/2} \sin^3 x \cos^5 x\,dx = \left.\frac{-\sin^2 x \cos^6 x}{8}\right]_0^{\pi/2} + \frac{2}{8}\int_0^{\pi/2} \sin x \cos^5 x\,dx$$

$$= \frac{2}{8}\int_0^{\pi/2} \sin x \cos^5 x\,dx = \frac{-2}{8}\left.\frac{\cos^6 x}{6}\right]_0^{\pi/2} = \frac{1}{24}$$

b) $$\int_0^{\pi/2} \sin^8 x \cos^2 x\,dx = \left.\frac{\cos x \sin^9 x}{10}\right]_0^{\pi/2} + \frac{1}{10}\int_0^{\pi/2} \sin^8 x\,dx$$

$$= \frac{1}{10}\int_0^{\pi/2} \sin^8 x\,dx$$

$$= \frac{1}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{7\pi}{512}$$

E8) a) $$L_0 = \int e^{ax}\,dx = \frac{e^{ax}}{a} + c$$

b) $L_1 = \int e^{ax} \sin x dx = \frac{e^{ax} \sin x}{a} - \frac{1}{a} \int e^{ax} \cos x dx$

$$= \frac{e^{ax} \sin x}{a} - \frac{e^{ax} \cos x}{a^2} - \frac{1}{a^2} \int e^{ax} \sin x dx$$

$$\therefore \int e^{ax} \sin x dx = \frac{ae^{ax} \sin x}{1+a^2} - \frac{e^{ax} \cos x}{1+a^2} + c$$

$$= \frac{e^{ax}}{1+a^2}(a \sin x - \cos x) + c$$

E9) $C_n = \frac{e^{ax}}{a} \cos^n x + \frac{n}{a} \int e^{ax} \cos^{n-1} x \sin x dx$

$$= \frac{e^{ax} \cos^n x}{a} + \frac{n}{a}\left[\frac{e^{ax} \cos^{n-1} x \sin x}{a} + \frac{1}{a} \int e^{ax} \{(n-1) \cos^{n-2} x \sin^2 x - \cos^n x\} dx\right]$$

$$= \frac{e^{ax} \cos^n x}{a} + \frac{n}{a^2} e^{ax} \cos^{n-1} x \sin x + \frac{n}{a^2} \int e^{ax} \{(n-1) \cos^{n-2} x - n \cos^n x\} dx$$

$$= \frac{e^{ax} \cos^n x}{a} + \frac{n}{a^2} e^{ax} \cos^{n-1} x \sin x + \frac{n(n-1)}{a^2} C_{n-2} - \frac{n^2}{a^2} C_n$$

$$\therefore C_n \frac{ae^{ax} \cos^n x}{n^2 + a^2} + \frac{ne^{ax} \cos^{n-1} x \sin x}{n^2 + a^2} + \frac{n(n-1)}{n^2 + a^2} C_{n-2}$$

E 10) Put $a = 0$ in the formula for C_n.

$$\therefore C_n = \int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$$

which is the reduction formula for $\int \cos^n x \, dx$

E 11) $\int \sinh^n x \, dx = \int \sinh^{n-1} x \sinh x \, dx$

$$= \sinh^{n-1} x \cosh x - (n-1) \int \sinh^{n-2} x \cosh^2 x dx$$

$$= \sinh^{n-1} x \cosh x - (n-1) \int \sinh^{n-2} x (1 + \sinh^2 x) dx$$

$$= \sinh^{n-1} x \cosh x - (n-1) I_{n-2} - (n-1) I_n$$

$$I_n = \frac{\sinh^{n-1} x \cosh x}{n} - \frac{n-1}{n} I_{n-2}$$

E 12) $I_n = \int \cosh^n x dx = \int \cosh^{n-1} x \cosh x dx$

$$= \cosh^{n-1} x \sinh x - (n-1) \int \cosh^{n-2} x \sinh^2 x \, dx$$

$$= \cosh^{n-1} x \sinh x - (n-1) \int \cosh^{n-2} x (\cosh^2 x - 1) dx$$

$$= \cosh^{n-1} x \sinh x - (n-1) I_n + (n-1) I_{n-2}$$

$$I_n = \frac{\cosh^{n-1} x \sinh x}{n} + \frac{n-1}{n} I_{n-2}$$